EUCLIDEAN GEOMETRY IN AN ORTHOGONAL FRAME

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1. Moving frames

In this note we describe the structure of Euclidean space using Élie Cartan's method of moving frames. More precisely, we wish to describe behaviour of a point as it moves in space. To do this, we will attach a frame to each point in space, which will serve as a "frame of reference" for the geometry around that point. The space of all configurations of a point together with a reference frame is given by,

$$Euc(n) = \left\{ \left(\begin{array}{c|c} 1 & * \\ \hline \mathbf{x} & A \end{array} \right) : \mathbf{x} \in \mathbb{R}^n \ , \ AA^t = Id \right\}.$$

Here, A represents an $n \times n$ orthogonal matrix satisfying the above relation. Also, $Euc(n)$ forms a group under matrix multiplication and is isomorphic to the (semi) direct product $\mathbb{R}^n \times O(n)$. Geometrically, $Euc(n)$ describes all rigid motions of the Euclidean space $\mathbb{R}^n$, i.e, it is the collection of all distance-preserving differentiable isomorphisms $\alpha : \mathbb{R}^n \to \mathbb{R}^n$. A map $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ is said to be distance-preserving if for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\alpha(\mathbf{x}) \cdot \alpha(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

We wish to measure how "geometric data" varies from one point to next, and write down differential equations that control this behaviour. While the calculation presented below work for any n, for the sake of simplicity we will restrict our attention to the three dimensional Euclidean space, $\mathbb{R}^3$.

Let M be a point in space. We consider an orthogonal frame at M given by three mutually perpendicular unit vectors I_1, I_2, I_3. This system of a point together with an orthogonal frame gives a trihedron $\mathcal{T} := (M, I_1, I_2, I_3)$ and can be thought as being a point of $Euc(3)$. In order to write down differential equations, we work in local coordinates of $Euc(3)$. It is important to note that there does not exist a global coordinate system on $Euc(3)$. This is because $Euc(3)$ contains rotations which cannot be described by a single continuous parameter.

The coordinate system we choose can be described around a point by six parameters $(u_1, \dots, u_6)$, representing the coordinates of M and the Euler angles defining the orthogonal frame (with respect to the standard axes). We will be interested in the behaviour of $\mathcal{T}$ as we vary the parameters u_i's. The object described by $\mathcal{T}$ with respect to the varying u_i's is what we call a *moving frame.*

2. The Maurer-Cartan Equations

Let $\mathcal{T}' = (M', I_1', I_2', I_3')$ be another trihedron which is infinitesimally close to $\mathcal{T}$, i.e,

$$\begin{aligned} M' - M =& \quad dM \\ I_k' - I_k =& \quad dI_k, \quad k = 1, 2, 3 \end{aligned}$$

Projecting the infinitesimal displacements dM and dI_k onto the axes I_k of $\mathcal{T}$, we get a system of differential equations,

$$dM = \omega^i I_i, \qquad dI_i = \omega_i^j I_j \tag{1}$$

where we use the summation convention on the index repeating above and below. Here, the ω^i, ω_i^j are differential 1-forms in the coordinates (u_i). Also, the vectors I_k's form an orthogonal frame, i.e, $I_i \cdot I_j = \delta_{ij}$. Differentiating we get, $dI_i \cdot I_j + I_i \cdot dI_j = 0$. This gives us another set of equations, $\omega_i^j + \omega_j^i = 0$. So, in fact, (1) is described by six differential forms, ω^i, ω_i^j. Thus, any moving frame satisfies (1) in terms of the six forms ω^i, ω_i^j. Conversely, one may ask,

Question 1. Given six forms ω^i, ω_i^j of an infinitesimal displacement of an orthonormal trihedron, does there exist a family of trihedrons admitting these components (as in (1))? Or to put it another way, when is such a system of 1-forms *integrable*?

To answer this we take the above system (1) and apply exterior differentiation,

$$dI_i \wedge \omega^i + I_i d\omega^i = 0$$
$$dI_j \wedge \omega_i^j + I_j d\omega_i^j = 0.$$

Substituting the values of dI_i's and rearranging terms, we get,

$$d\omega^i = \omega^j \wedge \omega_j^i, \qquad d\omega_i^j = \omega_i^k \wedge \omega_k^j \tag{2}$$

Hence, if a solution of (1) exists, it must satify (2). The Theorem of Cartan is that, in fact, this condition is also sufficient.

Theorem 2. [2, §26] If the differential forms ω^i, ω_i^j satisfy (2), then for a given initial trihedron $\mathcal{T}_0$ they define a moving frame obtained from the $\mathcal{T}_0$ by an appropriate affine transformation.

The equations (2) are also called the *Maurer-Cartan equations.*

3. Vistas

One can also extend this analysis to any manifold. In this setting, however, the system (2) of differential 2-forms may fail to hold, giving rise to non-trivial 2-forms,

$$d\omega^i - \omega^j \wedge \omega_j^i = \Omega^i, \qquad d\omega_i^j - \omega_i^k \wedge \omega_k^j = \Omega_i^j \tag{3}$$

The 2-forms Ω^i, Ω_i^j together describe the curvature of the manifold. In this sense, curvature of a manifold maybe thought of as the failure of a system of differential 1-forms to be integrable (as in Question 1). The components Ω^i describe the translation part of the curvature, whereas the Ω_i^j's describe the associated rotation (cf. [1, §36]).

For a Riemannian manifold, the translation part is zero by definition. This is because the Ω^i's describe the torsion tensor. Moreover, the Ω_i^j's describe the Riemann curvature tensor (cf. [2, Chapter 16]).

A moving frame, as described above, can be thought of as an integrable submanifold of the group $Euc(3)$. Moreover, the differential forms ω^i, ω_i^j, together describe a Lie algebra-valued 1-form ω with values in the Lie algebra of $Euc(3)$. Further, $Euc(3)$ admits a canonical left-invariant Lie algebra-valued 1-form which for any $g \in Euc(3)$ is given by $\omega_{Euc(3)} := g^{-1}dg$. This is known as the (left-invariant) Maurer-Cartan form of $Euc(3)$[1]. This point of view admits generalisation to any Lie group. In this language, Theorem 2 can be stated as,

Theorem 3. [3, Theorem 3.6.1] Let G be a Lie group with Lie algebra $\mathfrak{g}$. Denote by ω_G the Maurer-Cartan form of G. Let ω be a $\mathfrak{g}$-valued 1-form on a smooth manifold M satisfying the Maurer-Cartan equations. Then, for each point $p \in M$, there is a neighbourhood U of p and a smooth map $f : U \to G$ such that $f^*\omega_G = \omega$.

References

[1] Élie Cartan. Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie). *Annales scientifiques de l'École Normale Supérieure*, 3e série, 40:325–412, 1923.
[2] Élie Cartan. *Riemannian Geometry in an othrogonal frame.* World Scientific, 2001.
[3] R.W. Sharpe. *Differential Geometry.* Graduate Texts in Mathematics, no. 166. Springer, 1996.

[1]More generally for any Lie group G, the Maurer-Cartan form of G, is the $\mathfrak{g}$-valued form $\omega_G : T(G) \to \mathfrak{g}$ given by $\omega_G(v) = L_{g^{-1}*}(v)$, for any $v \in T_g(G)$. Here, L_g represents left multiplication by g. Every left-invariant differential form on G arises out of ω_G. For example, if $n = dim(G)$, $\bigwedge^n(\omega_G)$ is the Haar measure on G.